J. DIFFERENTIAL GEOMETRY
26 (1987) 327–335

A STRICT MAXIMUM PRINCIPLE FOR AREA MINIMIZING HYPERSURFACES

LEON SIMON

It is a well-known consequence of the Hopf maximum principle that if M_1, M_2 are smooth connected minimal hypersurfaces which are properly embedded in an open subset U of an $(n+1)$-dimensional Riemannian manifold N, if $\overline{M}_1 \sim M_1$, $\overline{M}_2 \sim M_2 \subset \partial U$, and if M_1 lies locally on one side of M_2 in a neighborhood of each common point, then either $M_1 = M_2$ or $M_1 \cap M_2 = \varnothing$.

If we replace the hypothesis that $\overline{M}_j \sim M_j \subset \partial U$ by the hypothesis that the $(n-1)$-dimensional Hausdorff measure (i.e. $\mathscr{H}^{n-1}$) of $\overline{M}_j \sim M_j \cap U$ vanishes for $j = 1, 2$, then we still have either $\overline{M}_1 = \overline{M}_2$ or $M_1 \cap M_2 = \varnothing$. However this latter alternative leaves open the question of whether or not $\overline{M}_1 \cap \overline{M}_2 \cap U = \varnothing$, and it is this question which interests us here.

Here we settle the question affirmatively in the *area minimizing* case. Specifically (in Theorem 1 of §1) we show that $\overline{M}_1 \cap \overline{M}_2 \cap U = \varnothing$ if $M_1 \cap M_2 = \varnothing$ in case M_1, M_2 are the regular sets (in U) of integer multiplicity currents T_1, T_2 which are mass minimizing in U and which have zero boundaries in U. (Notice that in this case we have automatically that $\mathscr{H}^{n-1}(\overline{M}_j \sim M_j \cap U) = 0$, $j = 1, 2$, by the regularity theory for codimension 1 currents.)

Our interest in this problem originated from the paper [1], and the question was again raised in [2, Problem 3.4]. The proof of the result (given in §2) depends rather heavily on the main results of [1].

1. Preliminaries and statement of main result

The optimal version of the main theorem concerns codimension 1 integer multiplicity locally rectifiable currents T (called simply "locally rectifiable" in [3] and henceforth simply called "integer multiplicity" here) which are mass minimizing in an open set U of the smooth $(n+1)$-dimensional oriented

Received February 20, 1986 and, in revised form, August 6, 1986.

Riemannian manifold N. Thus if W is open and $\overline{W}$ is a compact subset of U, then

$$\mathbf{M}_W(T) \leqslant \mathbf{M}_W(S)$$

for each integer multiplicity S with $\partial S = \partial T$ in U and $\operatorname{spt}(S - T) \subset\subset W$; here ∂S is the boundary of S in the sense of currents, $\operatorname{spt}(S - T)$ denotes the support of the current $S - T$, and $\mathbf{M}_W(S)$ is the mass of S in W ($= \sup S(\omega)$, where the sup is over smooth n-forms ω with compact support in W and with length $|\omega| \leqslant 1$ at each point of W). We shall actually be interested in the case when $\partial T = 0$ in U; i.e. when $T(d\omega) = 0$ for each smooth $(n - 1)$-form ω in N with support of $\omega \subset\subset U$.

We shall have occasion to use "oriented boundaries" in U; that is integer multiplicity (in fact multiplicity 1) currents T of the special form $T = (\partial[\![E]\!]) \llcorner U$, where E is an $\mathscr{H}^{n+1}$-measurable subset of N and $[\![E]\!]$ denotes the $(n + 1)$-dimensional current obtained by integration of $(n + 1)$-forms with compact support in N over the subset E. Actually if U is such that the n-dimensional integral homology of the pair $(N, N \sim U)$ is zero, then any integer multiplicity current T with $\partial T = 0$ in U can be decomposed (in U) into an $\mathbf{M}_U$-convergent sum ΣT_l of such oriented boundaries in such a way that $\mathbf{M}_U$ is additive (and hence so that each T_l is minimizing in U if T is minimizing in U). (See e.g. [3, 4.5] or [8, 27.8, 33.2].)

We shall also use the standard compactness and regularity theory for oriented boundaries which minimize mass in U; for this, and other standard facts about currents, we refer to e.g. [3], or [8, Chapters 6,7].

For any integer multiplicity T we let $\operatorname{reg} T$ (the regular set of T) be the set of points $\xi \in \operatorname{spt} T$ such that there is a neighborhood W of ξ in N with

$$T \llcorner W = k[\![M]\!],$$

where k is an integer and M is a smooth connected compact oriented embedded hypersurface in $\overline{W}$ with $\partial M \subset \partial W$ and with $\xi \in M$, and where $[\![M]\!]$ means the multiplicity 1 current obtained by integration of smooth n-forms (with compact support in N) over the hypersurface M. (Notice of course that $k = \pm 1$ in case T is an oriented boundary in U.) Also, we let

$$\operatorname{sing} T = \operatorname{spt} T \sim \operatorname{reg} T.$$

For $\lambda > 0$ we let (λ) denote the homothety of $\mathbb{R}^{n+1}$ taking x to λx.

We now state the main theorem:

Theorem 1. *Suppose T_1, T_2 are integer multiplicity currents with $\partial T_1 = \partial T_2 = 0$ in U, T_1, T_2 mass-minimizing in U, and $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 \cap U = \varnothing$. Then* $\operatorname{spt} T_1 \cap \operatorname{spt} T_2 \cap U = \varnothing$.

Remark 1. The main content of this theorem lies in the fact that $\operatorname{sing} T_1 \cap \operatorname{sing} T_2 \cap U = \varnothing$. Indeed previous work of Miranda ([7], also [8, 37.10]) establishes $\operatorname{sing} T_1 \cap \operatorname{reg} T_2 \cap U = \emptyset$. This latter result was recently shown to be true without the minimizing hypothesis by Solomon and White [9].

Remark 2. In case $N = \mathbb{R}^{n+1}$ and g is the standard Euclidean metric, Theorem 1 is straightforward to prove if $\operatorname{spt} T_1 \cap \operatorname{spt} T_2 \cap U$ is a priori assumed to be a compact subset of U, because in this case we can use a standard "cut-and-paste" argument (see e.g. [1], [6, 1.20], or [8, 37.10]) to show that $\operatorname{spt} T_1 \cap \operatorname{spt} T_2 \cap U = \emptyset$.

Using Theorem 1 we can establish the following corollary for oriented boundaries of least area.

Corollary 1. *Suppose* $T_1 = (\partial[\![E_1]\!]) \llcorner U$, $T_2 = (\partial[\![E_2]\!]) \llcorner U$ *are minimizing in* U, *with* $E_1 \cap U \subset E_2 \cap U$ *and with* $\operatorname{spt} T_1 \cap U$ *and* $\operatorname{spt} T_2 \cap U$ *connected. Then either* $T_1 = T_2$ *or* $\operatorname{spt} T_1 \cap \operatorname{spt} T_2 \cap U = \emptyset$.

Proof. Take an open geodesic ball $B_\rho(\xi) \subset U$ with ρ small enough to ensure that $\overline{B_\rho}(\xi)$ is diffeomorphic to the closed ball in $\mathbb{R}^{n+1}$, and let S_1, S_2 be components of $\operatorname{reg} T_1 \cap B_\rho(\xi)$, $\operatorname{reg} T_2 \cap B_\rho(\xi)$. Since $E_1 \subset E_2$ it follows that S_1 lies locally on one side of S_2 near each point of S_1. A well-known application of the Hopf maximum principle (see e.g. [6, pp. 103, 104]) then shows that $S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 = S_2$.

Next note that S_j, equipped with orientation from T_j, is minimizing in $B_\rho(\xi)$ and has zero boundary in $B_\rho(\xi)$ (see e.g. [8, 37.8]). Thus in the case $S_1 \cap S_2 = \emptyset$ we can apply Theorem 1 (with $U = B_\rho(\xi)$) to deduce that $\overline{S}_1 \cap \overline{S}_2 \cap B_\rho(\xi) = \emptyset$. On the other hand for any such components S_j which intersect $B_{\rho/2}(\xi)$ we have $\mathbf{M}(S_j) \geqslant c\rho^n$ (see e.g. [3, 5.1.6]), so at most finitely many components of $\operatorname{reg} T_j \cap B_\rho(\xi)$ can intersect $B_{\rho/2}(\xi)$.

Combining the above facts and using the given connectedness of $\operatorname{spt} T_1 \cap U$, $\operatorname{spt} T_2 \cap U$, the corollary now directly follows.

We now proceed to the proof of Theorem 1. We shall need the following lemma, which is an easy consequence of the regularity theorem for codimension 1 minimizing currents.

In this lemma we let $x = (x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1}$ be normal coordinates for N near x_0, with origin $x = 0$ corresponding to x_0 and with $T_{x_0}N$ identified with $\mathbb{R}^{n+1}$ via these coordinates in the usual way. The metric g is then $g_{ij}(x)\,dx^i dx^j$ with $g_{ij}(0) = \delta_{ij}$ and $\partial g_{ij}/\partial x^k(0) = 0$. We can take homotheties $(\lambda^{-1})_\# T$ ($\lambda > 0$) in terms of these local coordinates, and $(\lambda^{-1})_\# T$ is minimizing relative to the metric $g_{ij}(\lambda x)\,dx^i dx^j$ in case T is minimizing relative to g.

Lemma 1. *Let* $T = (\partial[\![E]\!]) \llcorner U$ *minimize in* U, $x_0 \in \operatorname{spt} T \cap U$, *and* ν *be the orienting unit normal for* T (*so* $*\nu = \vec{T}$), *and define* Ω_θ *to be the set of points* $x \in \operatorname{reg} T$ *which satisfy*

(i) $\operatorname{dist}(x, \operatorname{sing} T) > \theta|x|$ *and*

(ii) $\sup\{|x - y|^{-1}|\nu(x) - \nu(y)|\colon y \in \operatorname{reg} T,\ 0 < |y - x| < \theta|x|\} < (\theta|x|)^{-1}$.

Then there are $\rho_0 = \rho_0(x_0, T) > 0$ *and* $\theta_0 = \theta_0(x_0, T) > 0$ *such that* $\Omega_\theta \cap \partial B_\rho(x_0) \neq \emptyset$ $\forall \rho \in (0, \rho_0]$, $\theta \in (0, \theta_0]$.

Proof. If the lemma is false we can find a sequence $\rho_j \downarrow 0$ and

$$\left\{ x \in \operatorname{reg} T : |x| = \rho_j, \operatorname{dist}(x, \operatorname{sing} T) > j^{-1}\rho_j, \sup_{y \in \operatorname{reg} T, |x-y| < j^{-1}\rho_j} \left[|x-y|^{-1} |\nu(x) - \nu(y)| \right] < j\rho_j^{-1} \right\} = \emptyset. \tag{1}$$

Let $T_j = (\rho_j^{-1})_\# T$. From the existence of the tangent cones theorem (see e.g. [3, 5.4.3] or, [8, 37.4]) we know there is a subsequence $\{j'\}$ (henceforth denoted $\{j\}$) and a minimizing cone $C = \partial[\![F]\!]$ in $\mathbb{R}^{n+1}$ such that $T_j \to C$ (weak convergence of currents in $\mathbb{R}^{n+1}$), and $\operatorname{spt} T_j$ converges to $\operatorname{spt} C$ locally in the Hausdorff distance sense. By the De Giorgi-Allard regularity theorem this latter convergence is actually in the C^2 sense locally near points of $\operatorname{reg} C \neq \emptyset$. Thus $\exists y \in \operatorname{reg} C \cap S^n$, and we have fixed $\theta > 0$ and a sequence $y_j \in B_\theta(y) \cap \operatorname{reg} T_j \cap S^n$ with $y_j \to y$, $B_\theta(y) \cap \operatorname{spt} T_j \subset \operatorname{reg} T_j$, and

$$|x-z|^{-1} |\nu_j(x) - \nu_j(z)| \leqslant \theta^{-1} \text{ for } x, z \in B_\theta(y) \cap \operatorname{reg} T_j,\ x \neq z.$$

However in terms of the original T this means

$$B_{\theta\rho_j}(\rho_j y) \cap \operatorname{spt} T \subset \operatorname{reg} T$$

and

$$|x-z|^{-1} |\nu(x) - \nu(z)| \leqslant \theta^{-1}\rho_j^{-1} \quad \text{for } x, z \in B_{\theta\rho_j}(\rho_j y) \cap \operatorname{reg} T,\ x \neq z,$$

and since $\rho_j y_j \in \operatorname{reg} T \cap \partial B_{\rho_j}(0)$ and $y_j \to y$ this contradicts (1) for sufficiently large j.

2. Proof of Theorem 1

It suffices to consider the case when T_1, T_2 satisfy the additional hypotheses

$$T_1 = \partial[\![E_1]\!] \llcorner U, \quad T_2 = \partial[\![E_2]\!] \llcorner U, \qquad E_1 \subset E_2, \tag{$*$}$$

for some open $E_1, E_2 \subset U$. To see this, first note that we may assume (in view of the local nature of Theorem 1) that $\overline{U}$ is diffeomorphic to a ball in $\mathbb{R}^{n+1}$. Let S_j be a component of $\operatorname{reg} T_j \cap U$ equipped with a smooth orientation. Then (see e.g. [8, 37.8]) S_j is minimizing in U, $\partial S_j = 0$ in U, and (by the decomposition theorem [3, 4.5.17] or [8, 27.6]) we can write $S_j = \partial[\![E_j]\!] \llcorner U$ for some measurable $E_j \subset U$, $j = 1, 2$. Since the density of S_j is bounded below by 1 on $\overline{S}_j \cap U$, after alteration on a set of $\mathscr{H}^{n+1}$-measure zero we may take E_j to be a component of $U \sim \overline{S}_j$. (Part of the conclusion here is that there *is* more than one—in fact exactly 2—components of $U \sim \overline{S}_j$; this is of course a standard topological fact in case $\bar{S}_j \sim S_j \cap U = \emptyset$.) Notice that then E_j is connected because S_j is. Now let $K = \bar{S}_1 \cap \bar{S}_2 \cap U$ (so that $\mathscr{H}^{n-1}(K) = 0$ by the regularity theory, because $S_1 \cap S_2 = \emptyset$). By reversing orientations if necessary, we can arrange that $S_1 \cap E_2 \neq \emptyset$ and $S_2 \sim \bar{E}_1 \neq \emptyset$. Using the connectedness of S_1, S_2, and the Poincaré inequality [3, 4.5.3], together with the fact that $\mathscr{H}^{n-1}(K) = 0$, it then follows that $S_1 \subset E_2 \cup K$ and $S_2 \subset (U \sim \bar{E}_1) \cup K$. We claim it follows now that $E_1 \subset E_2$. Indeed otherwise (since E_1 is connected) we could choose a closed path γ in E_1 connecting a point in E_2 to a point in $\bar{S}_2$, thus showing $\bar{S}_2 \cap E_1 \neq \emptyset$, hence $S_2 \cap E_1 \neq \emptyset$, which contradicts the fact that $S_2 \subset (U \sim \bar{E}_1) \cup K \subset U \sim E_1$. Thus we have established that $S_1 = \partial[\![E_1]\!] \llcorner U$, $S_2 = \partial[\![E_2]\!] \llcorner U$ with $E_1 \subset E_2$. Since (cf. the argument in the proof of Corollary 1) at most finitely many components of $\operatorname{reg} T_1$, $\operatorname{reg} T_2$ can intersect a given compact subset of U, it now clearly follows that (by localizing and using suitable components S_1, S_2 as above) it is sufficient to consider only case $(*)$ of the theorem, as claimed.

We now suppose that we can find $x_0 \in \operatorname{spt} T_1 \cap \operatorname{spt} T_2 \cap U$, and we show that this leads to a contradiction. As in Lemma 1 we take normal coordinates $x = (x^1, \cdots, x^{n+1})$ for N with origin $x = 0$ corresponding to x_0 and with tangent space $T_{x_0}N$ identified with $\mathbb{R}^{n+1}$ via these coordinates. We can of course assume without loss of generality that U is contained in this coordinate neighborhood.

Still assuming $(*)$, we claim that we can reduce to the case when T_1, T_2 have the same tangent cones at the point x_0 $(= 0)$, in the strong sense that if $\{\lambda_j\}$ is *any* sequence $\downarrow 0$, then there is a subsequence $\{\lambda_{j'}\}$ (henceforth denoted $\{\lambda_j\}$) such that both $(\lambda_j^{-1})_{\#} T_1$ and $(\lambda_j^{-1})_{\#} T_2$ have the same cone as weak limit. Indeed suppose there is a sequence $\{\lambda_j\} \downarrow 0$ so that $(\lambda_j^{-1})_{\#} T_1$ and $(\lambda_j^{-1})_{\#} T_2$ converge to different cones $C_1 = \partial[\![F_1]\!]$ and $C_2 = \partial[\![F_2]\!]$ in $\mathbb{R}^{n+1}$. Since $E_1 \subset E_2$ we have $F_1 \subset F_2$ (up to a set of Lebesgue measure zero). We can now use the dimension reducing argument of [1] (appropriately modified) to give new $\tilde{T}_1$, $\tilde{T}_2$ satisfying the same hypotheses as T_1, T_2 (with $N = U = \mathbb{R}^{n+1}$), but having the same tangent cones at 0. To be precise, the dimension reducing argument of [1] goes as follows:

We can suppose C_1, C_2 (as above) contain a point $y \neq 0$ in the intersection of their supports by virtue of Remark 2. Then either C_1, C_2 have the same cones at y (in the strong sense) or else there are *distinct* tangent cones $D_1 = \partial[\![G_1]\!]$, $D_2 = \partial[\![G_2]\!]$ of C_1, C_2 at y with $G_1 \subset G_2$. But G_1, G_2 are cylinders $l \times E_1$, $l \times E_2$ (l the line containing y at 0), hence (after slicing with the hyperplane normal to l) we would have distinct $(n-1)$-dimensional minimizing currents $C_1 = \partial[\![E_1]\!]$, $C_2 = \partial[\![E_2]\!]$ in $\mathbb{R}^n$ with $E_1 \subset E_2$, $0 \in \operatorname{spt} C_1 \cap \operatorname{spt} C_2$. Next note that no such distinct C_1, C_2 can exist in case $n \leqslant 6$, because C_1, C_2 are hyperplanes if $n \leqslant 6$ by the regularity theory for codimension 1 minimizing currents (see e.g. [4] or [8, §37]). Thus the above arguments must (by induction on n) lead to a situation where we have distinct m-dimensional minimizing hypercones ($m > 6$) $\tilde{T}_1$, $\tilde{T}_2$, with $\tilde{T}_1 = \partial[\![H_1]\!]$, $\tilde{T}_2 = \partial[\![H_2]\!]$, with $H_1 \subset H_2$, and with a $y \in \operatorname{spt} \tilde{T}_1 \cap \operatorname{spt} \tilde{T}_2$ such that $\tilde{T}_1$, $\tilde{T}_2$ have the same tangent cones (in the strong sense) at y. Also by [1, Theorem 2] $\operatorname{reg} \tilde{T}_1$, $\operatorname{reg} \tilde{T}_2$ are connected. By an application of the Hopf maximum principle similar to that in Corollary 1 we can then also conclude $\operatorname{reg} \tilde{T}_1 \cap \operatorname{reg} \tilde{T}_2 = \varnothing$. Thus we may as well (and we shall) assume to begin with that T_1, T_2 have the same tangent cones at x_0. (Otherwise replace T_1, T_2, x_0 by $\tilde{T}_1$, $\tilde{T}_2$, y; notice that this does not upset the reduction (*).)

Now let $\rho_0, \theta_0, \Omega_\theta \subset \operatorname{reg} T_1$ be as in Lemma 1 with T_1 in place of T and define $h(x) = \operatorname{dist}(x, \operatorname{spt} T_2)$, $x \in \operatorname{spt} T_1$. Because T_1, T_2 have the same tangent cones at x_0 (in the strong sense), we know that, for each $\theta \leqslant \theta_0$, $r^{-1} \sup_{|x| = r, x \in \Omega_\theta} h(x) \to 0$ as $r \to 0$. In particular taking $\rho_j \downarrow 0$ such that

$$(1) \qquad \rho_j^{-1} \sup_{|x| = \rho_j, x \in \Omega_{\theta_0}} h(x) \geqslant \tfrac{1}{2} \rho^{-1} \sup_{|x| = \rho, x \in \Omega_{\theta_0}} h(x) \quad \text{for each } \rho \leqslant \rho_j,$$

we have that for each $\theta < 1$

$$(2) \qquad \sup_{x \in \Omega_{\theta_0}, |x| = \theta\rho_j} h(x) \leqslant 2\theta \sup_{x \in \Omega_{\theta_0}, |x| = \rho_j} h(x).$$

As in Lemma 1, there is a subsequence $\{j'\}$ (henceforth denoted $\{j\}$) such that $(\rho_j^{-1})_{\#} T_l \to C$, $l = 1,2$, and such that $\operatorname{spt}(\rho_j^{-1})_{\#} T_l$ converges locally in the Hausdorff distance sense in $\mathbb{R}^{n+1}$ to $\operatorname{spt} C$, $l = 1,2$. By the codimension 1 regularity theory (and in particular by the Allard-De Giorgi theorem—see e.g. [8, §37], [4]) and from the fact that $\operatorname{reg} C$ is connected [1, Theorem 2], we see that we can find C^2 functions $h_1^{(j)}$, $h_2^{(j)}$ defined over connected domains $U_j \subset \operatorname{reg} C$ such that

$$(3) \qquad \begin{aligned} &\left\{ x \in \operatorname{reg} C : \operatorname{dist}(x, \operatorname{sing} C) > \theta_j |x|,\ \theta_j < |x| < \theta_j^{-1} \right\} \subset U_j \quad \text{for some } \theta_j \downarrow 0, \\ &\lim_{j \to \infty} \left| h_l^{(j)} \right|_{C^2}^{*} = 0, \qquad l = 1,2 \end{aligned}$$

($|h|_{C^2}^{*} = \sup(|x|^{-1}|h(x)| + |\nabla h(x)| + |x||\nabla^2 h(x)|)$), and such that for each $\theta \in (0,1)$ and all $j \geqslant j(\theta)$ the following hold:

$$(4) \qquad \left\{ x \in \operatorname{reg}\left(\rho_j^{-1}\right)_{\#} T_l : \operatorname{dist}(x, \operatorname{sing} C) > \theta |x|,\ \theta < |x| < \theta^{-1} \right\} \subset G_l^{(j)} \subset \operatorname{reg}\left(\rho_j^{-1}\right)_{\#} T_l,$$

$$(5) \qquad \left(\rho_j^{-1}\right)(\Omega_{2\theta}) \cap \left\{ x : \theta < |x| < \theta^{-1} \right\} \subset \left\{ x \in \operatorname{reg}\left(\rho_j^{-1}\right)_{\#} T_1 : \operatorname{dist}(x, \operatorname{sing} C) > \theta |x| \right\}.$$

In (4), $G_l^{(j)}$ = graph of $h_l^{(j)} = H_l^{(j)}(U_j)$, where $H_l^{(j)}(x) = x + h_l^{(j)}(x)\nu(x)$, ν the unit normal of $\operatorname{reg} C$ pointing into F (recall $C = \partial[\![F]\!]$).

By (3), (4), (5), for any given $\theta \in (0,1)$ there are maps $p_j : (\rho_j^{-1})(\Omega_{2\theta}) \cap \{x : \theta < |x| < \theta^{-1}\} \to U_j$ with $H_1^{(j)}(p_j(x)) \, (= p_j(x) + h_1^{(j)}(p_j(x))\nu(p_j(x))) \equiv x$ and $\frac{1}{2}u_j(p_j(x)) \leqslant \rho_j^{-1}h(\rho_j x) \leqslant 2u_j(p_j(x))$ for all $x \in (\rho_j^{-1})(\Omega_{2\theta}) \cap \{x : \theta < |x| < \theta^{-1}\}$ and for all j sufficiently large, where h is as in (2) and where $u_j = h_1^{(j)} - h_2^{(j)}$ on U_j. (Since $u_j \neq 0$ ($\operatorname{reg} T_1 \cap \operatorname{reg} T_2 = \varnothing$), we may assume that $u_j > 0$ and U_j.) Then (2) implies

(6) $$\sup_{x \in (\rho_j^{-1})\Omega_{\theta_0}, |x| = \theta} u_j\big(p_j(x)\big) \leqslant 4\theta \sup_{x \in (\rho_j^{-1})\Omega_{\theta_0}, |x| = 1} u_j\big(p_j(x)\big)$$

for all sufficiently large j (depending on θ).

Now since $\operatorname{reg} T_1$, $\operatorname{reg} T_2$ are minimal hypersurfaces relative to the metric $g_{ij}(x)dx^i dx^j$, we know (by virtue (3) and (4)) that the difference $u_j = h_1^{(j)} - h_2^{(j)}$ satisfies an equation of the form

(7) $$\Delta_C u_j + |A_C|^2 u_j = \operatorname{div}\big(a_j \cdot \nabla u_j\big) + b_j \cdot \nabla u_j + c_j u_j,$$

with a_j, b_j, c_j converging uniformly to zero on compact subsets of $\operatorname{reg} C$; here Δ_C is the Laplacian on $\operatorname{reg} C$ and A_C is the second fundamental form of $\operatorname{reg} C$.

Since $u_j > 0$, by virtue of (7) and the connectedness of $\operatorname{reg} C$ we can use the Harnack inequality for divergence-form elliptic equations (in $\mathbb{R}^n$—see e.g. [5, §8.8]) to deduce

(8) $$\sup_K u_j \leqslant c_K \inf_K u_j, \qquad j \geqslant j(K),$$

for each compact $K \subset \operatorname{reg} C$. Hence the $C^{1,\alpha}$ Schauder theory (e.g. [5, §8.11]) tells us that

(9) $$|u_j|_{C^{1,\alpha}(K)} \leqslant c_K \inf_K u_j$$

for any compact $K \subset \operatorname{reg} C$ and for sufficiently large j (depending on K). Then letting y_0 be any fixed point of $\operatorname{reg} C$ we conclude there is a subsequence $\{u_{j'}\}$ (henceforth denoted $\{u_j\}$) such that $(u_j(y_0))^{-1}u_j$ converges locally in the C^1 sense on $\operatorname{reg} C$ to a positive solution u of

(10) $$\Delta_C u + |A_C|^2 u = 0$$

with $u(y_0) = 1$. In particular

(11) $$u > 0, \quad \Delta_C u \leqslant 0 \quad \text{on } \operatorname{reg} C.$$

We now want to apply the Harnack theory of [1] to u. Since C is minimizing, $\mathcal{H}^{n-2}(\operatorname{sing} C) = 0$ and $\mathbf{M}(C \llcorner B_\rho(y)) \leqslant c\rho^n$ $\forall \rho > 0$, $y \in \operatorname{spt} C$. Because of this it is easy to construct a sequence of functions $\{\varphi_j \subset C_c^\infty(\operatorname{reg} C)$

such that $\varphi_j \equiv 1$ on $\{x \in \operatorname{reg} C : j^{-1} < |x| < j, \operatorname{dist}(x, \operatorname{sing} C) > j^{-1}\}$, $0 \leqslant \varphi_j \leqslant 1$ everywhere on $\operatorname{reg} C$, and

$$\int_{\operatorname{reg} C \cap B_R(0)} |\nabla \varphi_j|^2 \to 0 \tag{12}$$

for each fixed $R > 0$. Now for $Q > 0$ let $u_Q = \min\{u, Q\}$, so that by (11) we have

$$\int_{\operatorname{reg} C} \nabla u_Q \cdot \nabla \zeta \geqslant 0 \tag{13}$$

for each nonnegative Lipschitz ζ with compact support in $\operatorname{reg} C$. Let $\psi \in C_c^\infty(\mathbb{R}^{n+1})$, $\psi_* = \psi \,|\operatorname{reg} C$, and replace ζ in (13) by $\varphi_j^2 \psi_*^2 u_Q^{-1}$. Then (13) gives

$$\int_{\operatorname{reg} C} u_Q^{-2} |\nabla u_Q|^2 \psi_*^2 \varphi_j^2 \leqslant c \int_{\operatorname{reg} C} \left(|\nabla \psi_*|^2 \varphi_j^2 + \psi_*^2 |\nabla \varphi_j|^2\right),$$

so that by (12) and the fact that $\varphi_j \to 1$ uniformly on compact subsets of $\operatorname{reg} C$, we have

$$\int_{B_R(0) \cap \operatorname{reg} C} |\nabla u_Q|^2 < \infty \quad \text{for each } R > 0,\ Q > 0. \tag{14}$$

Also, replacing ζ by $\varphi_j \psi_*$ in (13), and letting $j \uparrow \infty$, we have

$$\int_{\operatorname{reg} C} \nabla u_Q \cdot \nabla \psi_* \geqslant 0 \tag{15}$$

for each nonnegative $\psi \in C_c^\infty(\mathbb{R}^{n+1})$, where again $\psi_* = \psi \,|\operatorname{reg} C$.

In view of (14) and (15) we can indeed apply the Harnack theory of [1] in order to deduce that

$$\inf_{\operatorname{reg} C \cap B_2(0)} u_Q \geqslant c \int_{\operatorname{reg} C \cap B_2(0)} u_Q.$$

Letting $Q \uparrow \infty$ we thus have

$$\inf_{\operatorname{reg} C \cap B_2(0)} u \geqslant c \int_{\operatorname{reg} C \cap B_2(0)} u > 0.$$

In terms of the functions u_j this tells us in particular that for nonempty compact $L \subset \operatorname{reg} C \cap \overline{B}_{3/2}(0)$ there is $j_0 = j_0(L)$ such that

$$\inf_L u_j \geqslant c u_j(y_0) \quad \forall j \geqslant j_0,$$

where c is independent of L. Thus in view of (8) we deduce that there is $j_1 = j_1(K, L)$

$$\inf_L u_j \geqslant c_K \sup_K u_j \quad \forall j \geqslant j_1 \tag{16}$$

for any compact $L, K \subset \operatorname{reg} C \cap \overline{B}_{3/2}(0)$ with $L,\ K \neq \varnothing$, where $c_K > 0$ depends on K but not on L.

But now, taking $K = p_j((\rho_j^{-1})\Omega_{\theta_0} \cap \partial B_1)$ and $L = p_j((\rho_j^{-1})\Omega_{\theta_0} \cap \partial B_\theta)$, we see that (6), (16) are contradictory for sufficiently small θ. This completes the proof of Theorem 1.

References

[1] E. Bombieri & E. Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces*, Invent. Math. **15** (1972) 24–46.

[2] J. Brothers (Editor) *Open problems in geometric measure theory*, Proc. Sympos. Pure Math. (Arcata), Vol. 44, Amer. Math. Soc., Providence, RI, 1985, 369–377.

[3] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[4] ———, *Singular sets of area minimizing rectifiable currents with codimension* 1 *and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970) 767–771.

[5] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

[6] R. Hardt & L. Simon. *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985) 102–129.

[7] M. Miranda, *Sulle singularità della frontiere minimali*, Rend. Sem. Mat. Univ. Padova **38** (1967) 181–188.

[8] L. Simon, *Lectures on geometric measure theory*, Proc. Centre for Math. Analysis #3, Australian National University, 1983.

[9] B. Solomon & B. White, in preparation.

AUSTRALIAN NATIONAL UNIVERSITY